SKEW PRODUCTS OF BERNOULLI SHIFTS WITH ROTATIONS. II†

BY

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ABSTRACT

If T is a weakly mixing skew product transformation defined by $T(x,y) = (\sigma x, y + f(x) \pmod 1)$, where σ is a Bernoulli shift and f is a function satisfying a Hölder type condition and measurable with respect to the past of an independent partition of σ, then T is Bernoulli.

1. Introduction

We report here further results of our investigations of skew products of Bernoulli shifts with rotations [1]. We establish that if the family of rotations is measurable with respect to the past and satisfies a certain continuity condition then Kolmogorov implies Bernoulli. This paper is an application to a particularly simple situation of the geometric methods used to analyze geodesic flows on manifolds of negative curvature [6].

2. Skew products and main theorem

Let $(X, \mathscr{B}(X), \mu_X)$ denote the countable bilateral direct product of identical atomic measure spaces consisting of two points, say 0, 1, each having measure 1/2 and let σ be the shift transformation, that is, $(\sigma x)_n = x_{n+1}$, $-\infty < n < \infty$. Let $(Y, \mathscr{B}(Y), \mu_Y)$ denote the unit interval with Lebesgue measurable sets and Lebesgue measure and form the direct product

$$(Z, \mathscr{B}(Z), \mu_Z) = (X \times Y, \mathscr{B}(X \times Y), \mu_X \times \mu_Y).$$

Let f be a function from X into Y such that

† This work was partially supported by National Science Foundation under grant # GP 33581.
Received February 6, 1974 and in revised form April 23, 1974

(1) f is measurable with respect to the past, that is, f is a measurable function such that $f(x) = f(\bar{x})$ whenever $x_i = \bar{x}_i$ for all $i < 0$; and

(2) the series $\Sigma\,\alpha_n$ is convergent where

$$\alpha_n = \sup_{x,\bar{x}} \{|f(x) - f(\bar{x})| : x_{-1} = \bar{x}_{-1}, \cdots, x_{-n} = \bar{x}_{-n}\}.$$

Consider the family $\{\phi_x\}$ of rotations on Y defined by $\phi_x y = y + f(x) \pmod 1$. The transformation τ defined on Z by $\tau(x, y) = (\sigma x, \phi_x y)$ is called *the skew product* of the Bernouli shift σ with the family $\{\phi_x\}$ and is an invertible measure preserving transformation [2]. We shall finally assume that f is chosen so that

(3) τ is weak mixing (a transformation τ is said to be weak mixing (WM), if

$$\lim_{n\to\infty} \sum_{i=1}^{n} |\mu(\tau^i A \cap B) - \mu(A)\mu(B)| = 0$$

for all $A, B \in \mathcal{B}(Z)$).

It follows from a theorem of Parry [7] that τ will then be Kolmogorov (a definition of Kolmogorov will be stated later). Our goal is to prove the following theorem.

THEOREM. *τ is isomorphic to σ.*

3. Preliminaries

Our proof of this result is motivated by a representation of σ as the baker's transformation on the unit square [3, p. 9]. Geometrically, the transformation stretches the square to twice its length and half of its height, cuts it in two, and places the left half over the right. Analytically this is achieved by mapping the sequence x onto the point $(s, t) = (s(x), t(x))$ where the binary representation of s and t are given by

$$s = \cdot\, x_0 x_1 x_2 \cdots$$

$$t = \cdot\, x_{-1} x_{-2} x_{-3} \cdots .$$

This mapping sends μ_X onto Lebesgue measure of the unit square in the plane and carries σ over to the bakers' transformation given by

$$s(\sigma x) = \cdot\, x_1 x_2 \cdots = 2s \pmod 1$$

$$t(\sigma x) = \cdot\, x_0 x_1 \cdots = \begin{cases} t/2, & 0 \leqq s < 1/2 \\ (t-1)/2, & 1/2 \leqq s < 1. \end{cases}$$

The space Z can now be thought of as a cube with base X (the unit square as described above) and vertical side Y. This picture is particularly helpful, and with respect to it we define $H_{t,y}$ as a set of the form

$$H_{t,y} = \{(s,t,y) \mid 0 \leqq s \leqq 1\} \tag{4}$$

for $0 \leqq t < 1, 0 \leqq y < 1$, and call such a set a *horizontal past fiber.* Condition (1) implies that f as a function of (s,t) is constant for fixed t. Thus $\tau(H_{t,y})$ is the union of two past horizontal fibers with the same value of y, $\tau^2(H_{t,y})$ is the union of two pairs of horizontal fibers each pair possibly having different y-values, and $\tau^k(H_{t,y})$ is the union of 2^k such fibers, pairs of which having various y-values.

Let us review some of the basic facts about partitions and especially the concept of very weak Bernoulli which is the criterion from Ornstein's isomorphism theory which we use to obtain our theorem. For the purposes of this discussion let τ denote a general measure preserving transformation on a Lebesgue space $(Z, \mathcal{B}, \mu)$. A *finite partition* $\mathcal{P}$ is an ordered finite disjoint collection of measurable sets whose union is Z. Unless stated otherwise a partition is assumed to be finite. In the case of nondenumerable partitions the ordering is abandoned. The *distribution* of $\mathcal{P}$ is the n-tuple $(\mu(P_1), \cdots, \mu(P_n))$ where $P_1, \cdots, P_n$ are the elements of $\mathcal{P}$ in order. The *join* of two partitions $\mathcal{P}$ and $\mathcal{Q}$ is defined by

$$\mathcal{P} \vee \mathcal{Q} = \{P_i \cap Q_j \mid P_i \in \mathcal{P}, Q_j \in \mathcal{Q}\}$$

ordered lexicographically. Let us adopt the following convention. If $\{\mathcal{P}_i \mid i \in I\}$ is a family of partitions then $\mathbf{B}(\cup_{i \in I} \mathcal{P}_i)$ will denote the smallest σ-algebra of measurable sets containing all the atoms of all the $\mathcal{P}_i, i \in I$. If A is a set of positive measure then μ_A is defined as the normalized restriction of μ to A, that is $\mu_A(B) = \mu(A \cap B)/\mu(A)$; and $\mathcal{P}/A$, *the partition of A induced by* $\mathcal{P}$, that is, $\mathcal{P}/A = \{P_i \cap A \mid P_i \in \mathcal{P}\}$. Here the above convention can be relativized to A by defining $\mathbf{B}(\cup_{i \in I} \mathcal{P}_i/A)$ to mean the smallest σ-algebra of measurable subsets of A containing $\cup_{i \in I}\{P \cap A \mid P \in \mathcal{P}_i\}$. Two partitions $\mathcal{P}$ and $\mathcal{Q}$ are said to be *independent* if $\mu_{Q_j}(P_i) = \mu(P_i)$, $P_i \in \mathcal{P}$, $Q_j \in \mathcal{Q}$.

A partition $\mathcal{P}$ is called *Bernoulli* for τ if for each $n > 0$, the partitions $\mathcal{P}$ and $\vee_{-n}^{-1} \tau^i \mathcal{P}$ are independent. $\mathcal{P}$ is called a *generator* for τ if $\mathcal{B} = \mathbf{B}(\vee^{\infty} \mathcal{P})$ (for this definition $\mathcal{P}$ need not be finite). τ is called a *Bernoulli shift* if it has a Bernoulli generator.

Suppose $\mathcal{P}$ is a Bernoulli generator for τ. The Bernoulli condition says that for each atom $A \in \vee_{-m}^{-1} \tau^i \mathcal{P}$ the two partitions $\mathcal{P}/A$ and $\mathcal{P}$ have the same distribution. This will happen if and only if there is a measure preserving map $T_A : Z \to A$ (measure preserving in the sense that $\mu_A = \mu T_A^{-1}$), such that z and $T_A z$ lie in the same atom of $\mathcal{P}$. The concept of very weak Bernoulli is obtained by weakening this properly by requiring merely that most of the time $\tau^i(T_A z)$ and $\tau^i z$ lie in the same atom of $\mathcal{P}$ for most atoms A of $\vee_{-m}^{-1} \tau^i \mathcal{P}$ and most $z \in Z$. To be precise, a partition $\mathcal{P}$ is said to be *very weak Bernoulli* (VWB) *for* τ if for each $\epsilon > 0$, there exist an N and a set E with $\mu(E) < \epsilon$ such that for each $m > 0$ there exists a collection $\mathcal{C}$ of atoms of $\cup_{-m}^{-1} \tau^i \mathcal{P}$ satisfying $\mu(U\mathcal{C}) \geqq 1 - \epsilon$ such that if $A \in \mathcal{C}$ then there exists a measure preserving mapping $T_A : Z \to A$ such that if $n \geqq N$ and $z \in Z - E$ then $\tau^i(T_A z)$ and $\tau^i z$ lie in the same atom of $\mathcal{P}$ for at least $(1-\epsilon)n$ of the indices $i, 0 \leqq i \leqq n$.

In [4] Ornstein proved

(5) *if τ has a VWB generator then τ has a Bernoulli generator.* This theorem can be sharpened in two ways. First, it is not necessary to require in the definition of VWB that each T_A be measure preserving but merely that it be ϵ-almost measure preserving over certain sets, that is, for each $L \geqq 0$, there exists a family $\mathcal{C}' \subseteq \vee_{-m}^{L} \tau^i \mathcal{P}/A$ with $\mu_A(\cup \mathcal{C}') \geqq 1 - \epsilon$ such that if $B \in \mathcal{C}'$ then

$$|\mu_A(B)/\mu(T_A^{-1} B) - 1| < \epsilon.$$

This is proved in [6]. Second, the condition that τ have a VWB generator can be weakened to the condition that there is a sequence $\mathcal{P}^{(\nu)}$ of VWB partitions such that $\mathcal{P}^{(\nu+1)}$ refines $\mathcal{P}^{(\nu)}$ and the σ-algebra $\mathcal{B}$ is generated by the family of sets $\cup_{i,\nu} \tau^i \mathcal{P}^{(\nu)}$. This is proved in [5].

A weaker type of independence than Bernoulli for a partition is that of being Kolmogorov. A partition $\mathcal{P}$ is said to be *Kolmogorov* (K) if for any $B \in \boldsymbol{B}(\cup_{-\infty}^{\infty} \tau^i \mathcal{P})$ given $\epsilon > 0$, there is an $N_0 = N_0(\epsilon, B)$ such that for all $N' \geqq N \geqq N_0$ there is a family $\mathcal{C} \subseteq \vee_{N}^{N'} \tau^i \mathcal{P}$, $\mu(\cup \mathcal{C}) \geqq 1 - \epsilon$ such that if $A \in \mathcal{C}$

$$|\mu_A(B) - \mu(B)| < \epsilon.$$

A transformation τ is called K if it has a K-generator. A theorem due to Pinsker, Rokhlin, and Sinai [8] states that *if τ has a K-generator then every partition is K.*

4. Proof of the theorem

Let $\mathcal{H} = \{H_{t,y}\}$ be the (non-finite) partition of the cube Z into fibers (4). Let $\mathcal{P}^{(\nu)}$ be the partition $\{P_{j,k}^{(\nu)} \mid j, k = 0, \cdots, 2^\nu - 1\}$ in lexicographic order where

$$P_{j,k}^{(\nu)} = \{(s, t, j) \mid 0 \leqq s \leqq 1, \; j/2^\nu \leqq t < (j+1)/2^\nu, \; k/2^\nu \leqq y < (k+1)/2^\nu\}.$$

Such a set is a union of elements of $\mathcal{H}$. Clearly $\mathcal{P}^{(\nu)}$ increases in the sense of refinement to $\mathcal{H}$. $\mathcal{H}$ can be seen to be a generator by the following reasoning. By hypothesis f is nonconstant. Thus by continuity there are at least two small disjoint intervals of t each mapped by f into two distinct intervals. The second component of the points $\tau^n(s,t,y)$, $-\infty < n < 0$ visits each of the small t-intervals infinitely often for almost all points (s,t,y) of Z. This fact suffices to separate the fibers of $\mathcal{H} = \vee_0^\infty \tau^i \mathcal{H}$ into points, the elements of $\vee_{-\infty}^0 \tau^i \mathcal{H}$. The remarks following (5) therefore indicate that it is sufficient to establish

(6) Each $\mathcal{P}^{(\nu)}$ is VWB for τ.

Fix ν and let $\mathcal{P} = \mathcal{P}^{(\nu)}$. Summarizing the previous comments we must prove

(7) for each $\epsilon > 0$ there exists an N_0 and a set E for which $\mu(E) < \epsilon$ such that for each $m > 0$ there exists a collection $\mathcal{C} \subseteq \vee_{-m}^{-1} \tau^i \mathcal{P}$ satisfying $\mu(\cup \mathcal{C}) > 1-\epsilon$ such that for each $A \in \mathcal{C}$ there exists a measurable mapping $T_A : Z \to A$ such that first for each $L \geqq 0$ there is a family $\mathcal{C}' \subseteq \vee_{-m}^{L} \tau^i \mathcal{P}/A$ satisfying $\mu_A(\cup \mathcal{C}') \geqq 1-\epsilon$ such that if $B \in \mathcal{C}'$ then

$$|\mu_A(B)/\mu(T_A^{-1} B) - 1| < \epsilon$$

and second† if $N \geqq N_0$ and $z \in Z - E$ then $\tau^i(T_A z)$ and $\tau^i z$ lie in the same atom of $\mathcal{P}$ for at least $(1-\epsilon)N$ of indices i, $0 \leqq i \leqq N$.

To do this we first define a partition $\mathcal{Q} = \mathcal{Q}^{(r)}$ of Z into elements $Q_{j,k,l}$, $j,k,l = 0,\cdots,2^r-1$, called *mapping boxes*, that are defined by

$$Q_{j,k,l} = \{(s,t,y) \mid j/2^r \leqq s < (j+1)/2^r, k/2^r \leqq t < (k+1)/2^r, l/2^r \leqq y < (l+1)/2^r\}.$$

We shall see in a moment how to choose r. For each fixed $s \in [j/2^r, (j+1)/2^r)$ we define a set $Q_{j,k,l}(s)$ called an *s-section* of $Q_{j,k,l}$ by

$$Q_{j,k,l}(s) = \{(s,t,y) \mid k/2^r \leqq t < (k+1)/2^r, l/2^r \leqq y < (l+1)/2^r\}.$$

The binary expansion of each $t \in [k/2^r, k+1/2^r)$ begins with r terms $(x_0,\cdots,x_{r-1})$ where $\cdot x_0 \cdots x_{r-1} = k/2^r$. Now let δ be fixed and choose $M = M(\delta)$ such that

$$\sum_{n>M} \alpha_n < \delta, \tag{8}$$

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† Incidentally, we do not actually require the final condition concerning the indices i, $0 \leqq i \leqq N$ in the definition of VWB to hold for all $N \geqq N_0$. We just need it for $N = N_0$ [9, p. 92]. Furthermore the exceptional set E could be allowed to depend on A. Although these considerations improve the definition of VWB, we shall use the one as it stands above because our proof naturally obtains this stronger conclusion.

where $\{\alpha_n\}$ is given in (2). Now choose r such that $r > M$ and $1/2^r < \delta$.

For z and z' in $Q_{j,k,l}(s)$, the Euclidean distance $d(\tau^i z, \tau^i z')$ between corresponding points of the two orbits can be bounded for $i > 0$ as follows. From the triangle inequality

$$d(\tau^i z, \tau^i z') \leqq d(\tau^i(s,t,y), \tau^i(s,t,y')) + d(\tau^i(s,t,y'), \tau^i(s,t',y')).$$

By virtue of definition of τ

$$d(\tau^i(s,t,y), \tau^i(s,t,y')) < \delta$$

for all $i \geqq 0$. Next

$$d(\tau^i(s,t,y'), \tau^i(s,t',y')) \leqq d(\sigma^i(s,t), \sigma^i(s,t')) + \sum_{j=0}^{i-1} |f(\sigma^j(s,t)) - f(\sigma^j(s,t'))|.$$

Since (s,t,y') and (s,t',y') lie on a contracting fiber, the points $\tau^i(s,t,y')$ and $\tau^i(s,t',y')$ move together horizontally as i increases: thus

$$d(\sigma^i(s,t), \sigma^i(s,t')) < \delta$$

for all $i \geqq 0$. Finally by virtue of (8)

$$\sum_{j=0}^{i-1} |f(\sigma^j(s,t)) - f(\sigma^j(s,t'))| \leqq \sum_{n>r} \alpha_n < \delta$$

for all $i \geqq 0$. Thus

$$d(\tau^i z, \tau^i z') \leqq 3\delta \tag{9}$$

for all $z, z' \in Q_{j,k,l}(s)$ and $i \geqq 0$.

For any $m > l$ and any atom $A \in \vee_{-m}^{l} \tau^i \mathcal{P}$ let us proceed to define $T_A : Z \to A$. Let Q denote one of the mapping boxes $Q_{j,k,l}$ and $Q(s_0)$ its first s-section where $s_0 = j/2^r$. Let π be the projection of Q along past horizontal fibers onto $Q(s_0)$, i.e., $\pi(s,t,y) = (s_0, t, y)$ for $(s,t,y) \in Q$. Let us assume that n and A are such that $\mu(\tau^n A \cap Q) > 0$ for all $Q \in \mathcal{Q}$. That it is admissable to make this assumption, follows from the Kolmogorov condition and the finiteness of Q, (15) below. Since a horizontal fiber traverses completely any Q it touches and since any $B \in \boldsymbol{B}(\vee_{n}^{\infty}{}_{m} \tau^i \mathcal{P})$, and this includes $\tau^n A$ as well, is a union of past horizontal fibers, it follows that $(Q(s), \boldsymbol{B}(\cup_{n}^{\infty}{}_{m} \tau^i \mathcal{P}/Q(s), \mu_{Q(s)})$ and $(\tau^n A \cap Q(s), \boldsymbol{B}(\cup_{n}^{\infty}{}_{m} \tau^i \mathcal{P}/r^n A \cap Q(s)), \mu_{\tau^n A \cap Q(s)}$ are Lebesgue spaces where $\mu_{Q(s)}$ and $\mu_{\tau^n A \cap Q(s)}$ are defined by the relations:

$$\mu_{Q(s)}(B) = \mu_Q(B) \tag{10}$$

$$\mu_{\tau^n A \cap Q(s)}(B) = \mu_{\tau^n A \cap Q}(B).$$

Let

$$S:\left(Q(s_0),\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}/Q(s_0)\right),\ \mu_{Q(s_0)}\right)\rightarrow\left(\tau^n A\cap Q(s_0),\right.$$

$$\left.\boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\tau^i\mathcal{P}/\tau^n A\cap Q(s_0)\right),\ \mu_{\tau^n A\cap Q(s_0)}\right)$$

be any one of the multitude of isomorphisms between these two Lebesgue spaces. Using again the same symbol S we have a unique isomorphism

$$S:\left(Q(s),\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}/Q(s)\right),\ \mu_{Q(s)}\right)\rightarrow\left(\tau^n A\cap Q(s),\right.$$

$$\left.\boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}/\tau^n A\cap Q(s)\right),\ \mu_{\tau^n A\cap Q(s)}\right)$$

for each $Q(s)\subseteq Q$ that satisfies

$$S\pi=\pi S.$$

These mappings naturally define an isomorphism

$$S:\left(Q,\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}/Q\right),\ \mu_Q\right)\rightarrow\left(\tau^n A\cap Q,\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}/\tau^n A\cap Q\right),\ \mu_{\tau^n A\cap Q(s)}\right)$$

and a measureable mapping

$$S:\left(Z,\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}\right)\right)\rightarrow\left(\tau^n A,\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}/\tau^n A\right)\right).$$

The key facts to keep in mind about S are that it preserves sections, i.e.,

(11) $$SQ(s)\subset Q(s),$$

and

(12) $$\mu_Q(S^{-1}B)=\mu_{\tau^n A\cap Q}(B)$$

for any $B\in\boldsymbol{B}(\cup_{n-m}^{\infty}\tau^i\mathcal{P})$. We define

$$T_A:\left(Z,\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\ \tau^i\mathcal{P}\right)\right)\rightarrow\left(A,\ \boldsymbol{B}\left(\bigcup_{n-m}^{\infty}\tau^i\mathcal{P}/A\right)\right)$$

by

(13) $$T_A=\tau^{-n}S\tau^n.$$

From (12) and (13) we have for any $B\in\boldsymbol{B}(\cup_{-m}^{\infty}\tau^i\mathcal{P}/A)$, $\mu_A(B)>0$, that

$$(14)\quad |\mu(T_A^{-1}B)/\mu_A(B)-1| = |\mu(S^{-1}\tau^n B)/\mu_{\tau^n A}(\tau^n B)-1|$$

$$= \left|\sum_{Q\in\mathcal{Q}} [\mu(Q)\mu_Q(S^{-1}\tau^n B)/\mu_{\tau^n A}(\tau^n B)-\mu(Q)]\right|$$

$$= \left|\sum_{Q\in\mathcal{Q}} [\mu(Q)\mu_{\tau^n A\cap Q}(\tau^n B)/\mu_{\tau^n A}(T^n B)-\mu(Q)]\right|$$

$$\leqq \sum_{Q\in\mathcal{Q}} (\mu(Q)/\mu_{\tau^n A}(Q))\,|\,\mu_{\tau^n B}(Q)-\mu_{\tau^n A}(Q)\,|.$$

The moment has come to choose n and the families $\mathscr{C}$ and $\mathscr{C}'$ of (7). We shall first work with δ instead of ϵ. Fix $\delta > 0$. From the Kolmogorov† condition and the finiteness of Q, we can choose an integer n_0 so that for each $m \geqq 1$, there is a class $\mathscr{C}_m \subseteq \vee_{-m}^{-1}\tau^i\mathscr{P}$, with $\mu(\cup\mathscr{C}_m)\geqq 1-\delta$, such that if $n \geqq n_0$ and $A \in \mathscr{C}_m$ then

$$(15)\quad |\mu_{\tau^n A}(Q)-\mu(Q)| < \delta$$

for every $Q \in \mathcal{Q}$. Furthermore, there is an n_1 such that for each $m \geqq 0$, and $A \in \mathscr{C}_m$ there exists a family $\mathscr{C}' \subseteq \vee_{-m}^{L}\tau^i\mathscr{P}/A$, with $\mu_A(\cup\mathscr{C}')\geqq 1-\delta$, such that if $n \geqq n_1$ and $B \in \mathscr{C}'$ then

$$(16)\quad |\mu_{\tau^n B}(Q)-\mu(Q)| < \delta$$

for every $Q \in \mathcal{Q}$. Thus choosing $n > \max(n_0, n_1)$ in the definition of S it follows from (14) that

$$(18)\quad |\mu(T_A^{-1}B)/\mu_A(B)-1| \leqq 2\delta \sum_{Q\in\mathcal{Q}} \mu(Q)^2/(\mu(Q)-\delta)$$

for $B \in \mathscr{C}^1$.

Finally we are ready to produce N_0 and E of (7). For the partition $\mathscr{P}$ we can use the ergodic theorem in combination with Egoroff's theorem to find a set $E, \mu(E) < \delta$, and an integer N_1 so that if $z \in E$ and $N \geqq N_1$ then $\tau^i z$ is within 3δ of the top and bottom boundary of any set of $\mathscr{P}$ for at most $N\delta(1+6\cdot 2^\nu)$ of the indices $i, 0 \leqq i \leqq N$. (Note the measure of the set of points within 3δ of the top and bottom of any set in $\mathscr{P}$ is $6\cdot 2^\nu\delta$). From (9) and (11) $d(\tau^i z, \tau^i T_A z) < 3\delta$ for $i \geqq n$ and all $z \in Z$. Thus if $z \in Z - E, \tau^i z$ and $\tau^i(T_A z)$ will lie in the same set in $\mathscr{P}$ for at least $N(1-\delta(1+6\cdot 2^\nu))-n$ of the indices

† At this stage careless reasoning was employed in theorem 2 [1]. Mixing was used, and mixing is not strong enough to yield (16). Nevertheless, the present material repairs the difficulty.

$i, 0 \leqq i \leqq N$. Now let $N_0 = \max(N_1, n/\delta)$. For $z \in Z - E$ and $N \geqq N_0$ we have $\tau^i z$ and $\tau^i(T_A z)$ lying in the same set of $\mathcal{P}$ for at least $N(1 - \delta(2 + 6 \cdot 2^\nu))$ of the indices $i, 0 \leqq i \leqq N$.

For any $\epsilon > 0$ we choose δ small enough to satisfy simultaneously

$$\delta < \epsilon \tag{19}$$

$$2\delta \sum_{Q \in \mathcal{P}} \mu(Q)^2/(\mu(Q) - \delta) < \epsilon \tag{20}$$

$$\delta(2 + 6 \cdot 2^\nu) < \epsilon \ . \tag{21}$$

The existence of the required items in (7) now follows and the theorem is proved.

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